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On an isoperimetric inequality for infinite finitely generated groups

Andrzej Żuk*

CNRS, Ecole Normale Supérieure de Lyon, Unité de Mathématiques Pures et Appliquées, 46, Allée d'Italie, F-69364 Lyon cedex 07 France

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Abstract

Let Γ be an infinite, finitely generated group. We prove that for any finite subset A of Γ the following inequality is true:

$$|A| \leqslant \sum_{\gamma \in \partial A} \operatorname{dist}(e, \gamma),$$

where $\operatorname{dist}(e,\gamma)$ is a distance in Γ of γ to the identity element e, and ∂A is a boundary of A. This inequality implies that the volume form on the universal cover of a compact Riemannian manifold with infinite fundamental group has a primitive of at most linear growth. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Let Γ be an infinite discrete group generated by a finite subset S, which we suppose to be symmetric, i.e. $S = S^{-1}$. For a finite subset A of Γ let $|A|$ be the number of its elements and ∂A its boundary, i.e.

$$\partial A = \{\gamma \notin A;\ \text{there exists } s \in S \text{ such that } s\gamma \in A\}.$$

* Tel.: + 33-4-72-72-83-69; fax: + 33-4-72-72-84-80.
E-mail address: azuk@umpa.ens-lyon.fr (A. Żuk)

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Let e be the identity element in Γ and let $\operatorname{dist}(e,\gamma)$ denote the distance from γ to e with respect to the set S, i.e. the minimal number of elements from S needed to represent γ. The distance between γ and γ' is defined as $\operatorname{dist}(\gamma,\gamma') = \operatorname{dist}(e, \gamma\cdot\gamma'^{-1})$, i.e. we consider a right invariant distance.

The aim of this paper is to prove the following isoperimetric inequality, conjectured by Sikorav [6].

Theorem 1. *For any finite subset A of Γ the following holds*:

$$|A| \leqslant \sum_{\gamma\in\partial A} \operatorname{dist}(e,\gamma). \tag{1}$$

Remark. (1) Up to a positive constant, inequality (1) is trivially satisfied for non-amenable groups (see [2]).

(2) Because of Γ-invariance, inequality (1) implies that for any $\gamma_0 \in \Gamma$ we have

$$|A| \leqslant \sum_{\gamma\in\partial A} \operatorname{dist}(\gamma_0,\gamma).$$

But for $\gamma_0 \in A$ and $\gamma \in \partial A$ we have $\operatorname{dist}(\gamma_0,\gamma) \leqslant \operatorname{diam}(A) + 1$, where $\operatorname{diam}(A)$ is the diameter of the set A. Thus Theorem 1 implies the following isoperimetric inequality:

$$|A| \leqslant (\operatorname{diam}(A) + 1)\,|\partial A|, \tag{2}$$

where $|\partial A|$ is the number of elements in its boundary. The inequality (2) was proved in [1].

1.1. The idea behind the proof

Now, we would like to present the idea behind the proof which will be expanded upon in later sections.

Let us consider any geodesic g in the group Γ. By the triangular inequality one has

$$|g\cap A| \leqslant \sum_{\gamma\in(\partial A\cap g)} \operatorname{dist}(\gamma, e), \tag{3}$$

where $|g\cap A|$ is the number of intersection points of g with A (see Fig. 1). Now we will show, that if we choose an invariant measure μ on a non-empty set of geodesics $\mathscr{G}$, $\sum_{g\in\mathscr{G}} |g\cap A|\,\mu(g)$ is equal to $|A|$

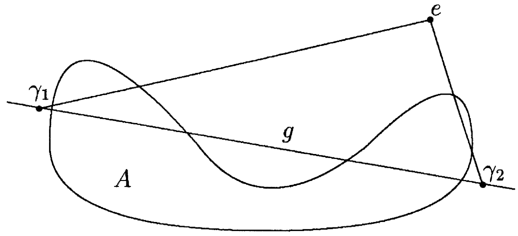

Fig. 1. The intersection of the geodesic g with the set A.

and $\sum_{g\in\mathscr{G}}\sum_{\gamma\in(\partial A\cap g)}\mathrm{dist}(\gamma,e)\mu(g)$ is equal to $\sum_{\gamma\in\partial A}\mathrm{dist}(\gamma,e)$. Thus the isoperimetric inequality (1) follows from the inequality (3).

The proof of the isoperimetric inequality is given in Section 2 and the measure μ is constructed in Section 3. Finally in Section 4, we mention some geometric consequences of Theorem 1.

2. The proof of the isoperimetric inequality

Let $\mathscr{G}$ be any non-empty subset of bi-infinite (parametrized or not) geodesics in Γ which is Γ-invariant. Let $G\subset\mathscr{G}$ be the set of geodesics of $\mathscr{G}$ containing e.

Let $\mathscr{F}$ be the right-invariant family of those subsets $B\subset\mathscr{G}$ which are covered by a finite number of translations of G (these subsets we call *measurable*), i.e. for $B\subset\mathscr{G}$, the subset B belongs to $\mathscr{F}$ if there exist $\gamma_1,\dots,\gamma_n\in\Gamma$ such that

$$B\subset G\gamma_1\cup G\gamma_2\cup\cdots\cup G\gamma_n.$$

In order to prove Theorem 1 we need a finitely additive measure μ on $\mathscr{F}$, satisfying the following conditions:

(i) μ is right invariant, i.e. $\mu(B\gamma)=\mu(B)$ for $B\in\mathscr{F}$ and $\gamma\in\Gamma$,
(ii) μ is finitely additive, i.e. $\mu(B_1\cup B_2)=\mu(B_1)+\mu(B_2)-\mu(B_1\cap B_2)$ for $B_1,B_2\in\mathscr{F}$, and $\mu(G)=1$.
An example of the triple $(\mathscr{G},\mathscr{F},\mu)$ is constructed in Section 3. Using any triple $(\mathscr{G},\mathscr{F},\mu)$ we can prove Theorem 1.

Proof of Theorem 1. Let J be the set of all possible intersections of geodesics from $\mathscr{G}$ with the subset $A\subset\Gamma$ and its boundary, i.e.

$$J=\{g\cap(A\cup\partial A);g\in\mathscr{G}\}.$$

This set is finite. For a given intersection $I\in J$, let G_I be the set of geodesics in $\mathscr{G}$ whose intersection with $A\cup\partial A$ is I, i.e.

$$G_I=\{g\in\mathscr{G};g\cap(A\cup\partial A)=I\}.$$

The sets G_I are measurable, i.e. $G_I\in\mathscr{F}$. For $I_1\neq I_2\in J$ we have

$$G_{I_1}\cap G_{I_2}=\emptyset$$

and for a given $\gamma\in(A\cup\partial A)$ one has

$$\bigcup_{I\in J,\gamma\in I}G_I=G\gamma.$$

Thus for $\gamma \in (A \cup \partial A)$ by the properties of the measure μ one has

$$\sum_{I \in J,\ \gamma \in I} \mu(G_I) = \mu(G\gamma) = 1. \tag{4}$$

Let $|I \cap A|$ denote the number of elements of I which are in A. Then from (4) one has

$$\begin{aligned}\sum_{I \in J} |I \cap A| \mu(G_I) &= \sum_{I \in J} \sum_{\gamma \in A, \gamma \in I} \mu(G_I) \\ &= \sum_{\gamma \in A} \sum_{I \in J, \gamma \in I} \mu(G_I) \\ &= \sum_{\gamma \in A} 1 \\ &= |A|.\end{aligned}$$

On the other hand, from (4) one also has

$$\begin{aligned}\sum_{I \in J} \sum_{\gamma \in (I \cap \partial A)} \operatorname{dist}(e, \gamma) \mu(G_I) &= \sum_{\gamma \in \partial A} \operatorname{dist}(e, \gamma) \sum_{I \in J, \gamma \in I} \mu(G_I) \\ &= \sum_{\gamma \in \partial A} \operatorname{dist}(e, \gamma).\end{aligned}$$

Thus in order to prove the isoperimetric inequality (1) we need only prove that

$$|A \cap I| \leqslant \sum_{\gamma \in (\partial A \cap I)} \operatorname{dist}(e, \gamma).$$

If I has an empty intersection with A then this is clear. Otherwise let us consider a geodesic $g \in \mathscr{G}$ whose intersection with $A \cup \partial A$ is I, i.e. $I = g \cap (A \cup \partial A)$.

Let γ_1 and γ_2 be the first and the last points of intersection of g with ∂A (see Fig. 1). They exist and are different because g has a non-empty intersection with A. Thus,

$$|A \cap I| \leqslant \operatorname{dist}(\gamma_1, \gamma_2) \leqslant \operatorname{dist}(\gamma_1, e) + \operatorname{dist}(\gamma_2, e) \leqslant \sum_{\gamma \in (\partial A \cap I)} \operatorname{dist}(\gamma, e),$$

which ends the proof of the isoperimetric inequality. $\square$

3. The construction of a measure on a subset of geodesics

In this section we first construct a subset of parametrized geodesics and then construct a finitely additive measure on a certain family of its subsets.

3.1. A subset of geodesics

We will be interested in bi-infinite geodesics, so first we prove the following:

Lemma 1. *For each element γ of Γ there is at least one bi-infinite geodesic which passes through γ.*

Proof. We can suppose that $\gamma = e$. First of all we prove that there exists at least one geodesic starting at e which is infinite in one direction. This is because for each $n \in \mathbb{N}$ we can construct a geodesic of length n starting at e, and in order to have an infinite geodesic l, we can use the diagonal extraction argument (the group Γ is finitely generated). Now, using translations of l we can obtain a sequence of geodesics l_n which pass through e, such that on one side of e the geodesic l_n is infinite and on the other side its length is n. Using the diagonal extraction argument again we can obtain a bi-infinite geodesic which passes through e. $\square$

Now let g_0 be one of the bi-infinite geodesics which passes through e.

The set of geodesics $\mathscr{G}$ that we want to consider consists of parameterized geodesics which are right translations of g_0 by the elements of Γ, i.e.

$$\mathscr{G} = \{g_0\gamma;\ \gamma \in \Gamma\}.$$

Because we are considering parameterized geodesics, for $\gamma \neq \gamma'$ we have

$$g_0\gamma \neq g_0\gamma'. \tag{5}$$

In the next subsection, we will consider the sets G and $\mathscr{F}$ associated to $\mathscr{G}$, and define a measure μ on $\mathscr{F}$ satisfying (i) and (ii) as in Section 2.

3.2. The construction of a measure

Let m be a finitely additive measure on all subsets of $\mathbb{Z}$ which is invariant by translations and such that $m(\mathbb{Z}) = 1$ (for the existence of such a measure see for instance [3]). We use the measure m to construct the measure μ.

The geodesic g_0 is a sequence $\{\gamma_i\}_{i\in\mathbb{Z}}$ of elements of Γ such that

$$\operatorname{dist}(\gamma_i, \gamma_j) = |i - j|.$$

Thus there is a natural bijection $\psi : g_0 \to \mathbb{Z}$ between elements of g_0 and integers given by

$$\psi(\gamma_i) = i$$

for $\gamma_i \in g_0$.

Because we are considering parameterized geodesics, inequality (5) holds and each geodesic in $\mathscr{G}$ can be represented by a unique element in Γ. All elements of Γ which correspond to G are in a natural correspondence with elements of g_0, i.e.

$$G = \{g_0\gamma^{-1}\colon \gamma \in g_0\}.$$

Thus one can define a natural bijection $\Psi: G \to \mathbb{Z}$ between elements of G and integers given by

$$\Psi(g_0\gamma^{-1}) = \psi(\gamma)$$

for $g_0\gamma^{-1} \in G$.

This correspondence can be naturally extended to $G\gamma' = \{g_0\gamma^{-1}\gamma' : \gamma \in g_0\}$, i.e. there is a natural bijection $\Psi_{\gamma'}: G\gamma' \to \mathbb{Z}$ between elements of $G\gamma'$ and the integers given by

$$\Psi_{\gamma'}(g_0\gamma^{-1}\gamma') = \psi(\gamma)$$

for $g_0\gamma^{-1}\gamma' \in G\gamma'$. In particular $\Psi_e = \Psi$.

The bijection $\Psi_{\gamma'}$ and the measure m enable us to define a measure $m_{G\gamma'}$ for every subset $B \subset G\gamma'$. Namely, for $B \subset G\gamma'$ we define

$$m_{G\gamma'}(B) = m(\Psi_{\gamma'}(B)).$$

Now we define the measure μ. For $B \in \mathscr{F}$, i.e. $B \subset G\gamma_1 \cup \cdots \cup G\gamma_n$ the measure μ is defined in the following way:

$$\begin{aligned}\mu(B) = {} & m_{G\gamma_1}(B \cap G\gamma_1) + m_{G\gamma_2}((B \cap G\gamma_2) \backslash G\gamma_1) + \cdots \\ & + m_{G\gamma_i}((B \cap G\gamma_i) \backslash (G\gamma_1 \cup \cdots \cup G\gamma_{i-1})) + \cdots \\ & + m_{G\gamma_n}((B \cap G\gamma_n) \backslash (G\gamma_1 \cup \cdots \cup G\gamma_{n-1})).\end{aligned}$$

All properties of μ (right Γ-invariance, finite additivity and $\mu(G) = 1$) are clear as soon as we prove that μ is well defined. Indeed, finite additivity of μ follows from finite additivity of measures $m_{G\gamma_i}$ and $\mu(G) = m_G(G) = m(\mathbb{Z}) = 1$. Finally, we have $B\gamma = G\gamma_1\gamma \cup \cdots \cup G\gamma_n\gamma$ and thus

$$\begin{aligned}\mu(B\gamma) &= m_{G\gamma_1\gamma}(B\gamma \cap G\gamma_1\gamma) + \cdots + m_{G\gamma_n\gamma}((B\gamma \cap G\gamma_n\gamma) \backslash (G\gamma_1\gamma \cup \cdots \cup G\gamma_n\gamma)) \\ &= m_{G\gamma_1}(B \cap G\gamma_1) + \cdots + m_{G\gamma_n}((B \cap G\gamma_n) \backslash (G\gamma_1 \cup \cdots \cup G\gamma_n)) \\ &= \mu(B)\end{aligned}$$

which proves right Γ-invariance of μ. Thus we need to show

Proposition 1. *The measure μ is well defined, i.e. its definition does not depend on the covering or its order.*

Proof. First of all we note that we need only prove that the definition does not depend on the order of the covering. Indeed if we have two coverings

$$B \subset G\gamma_{i_1} \cup \cdots \cup G\gamma_{i_n} \tag{6}$$

and

$$B \subset G\gamma_{j_1} \cup \cdots \cup G\gamma_{j_k} \tag{7}$$

one can consider the two coverings

$$B \subset G\gamma_{i_1} \cup \cdots \cup G\gamma_{i_n} \cup G\gamma_{j_1} \cup \cdots \cup G\gamma_{j_k}, \tag{8}$$

$$B \subset G\gamma_{j_1} \cup \cdots \cup G\gamma_{j_k} \cup G\gamma_{i_1} \cup \cdots \cup G\gamma_{i_n}. \tag{9}$$

Thus the definition of $\mu(B)$ is the same for coverings (6) and (8) by construction. And similarly the definition of $\mu(B)$ is the same for coverings (7) and (9). But coverings (8) and (9) are different from each other only by their order.

Now any permutation can be represented by neighboring transpositions. Thus in order to prove that μ is well defined it is enough to prove that the definition of $\mu(B)$ is the same for the two coverings

$$B \subset G\gamma_1 \cup \cdots \cup G\gamma_{i-1} \cup G\gamma_i \cup G\gamma_{i+1} \cup G\gamma_{i+2} \cup \cdots \cup G\gamma_n, \tag{10}$$

$$B \subset G\gamma_1 \cup \cdots \cup G\gamma_{i-1} \cup G\gamma_{i+1} \cup G\gamma_i \cup G\gamma_{i+2} \cup \cdots \cup G\gamma_{i_n}. \tag{11}$$

By definition of $\mu(B)$ with respect to coverings (10) and (11) we have to show that

$$\begin{aligned} &m_{G\gamma_i}((B \cap G\gamma_i) \backslash (G\gamma_1 \cup \cdots \cup G\gamma_{i-1})) + m_{G\gamma_{i+1}}((B \cap G\gamma_{i+1}) \backslash (G\gamma_1 \cup \cdots \cup G\gamma_{i-1} \cup G\gamma_i)) \\ &\quad = m_{G\gamma_{i+1}}((B \cap G\gamma_{i+1}) \backslash (G\gamma_1 \cup \cdots \cup G\gamma_{i-1})) + m_{G\gamma_i}((B \cap G\gamma_i) \backslash (G\gamma_1 \cup \cdots \cup G\gamma_{i-1} \cup G\gamma_{i+1})). \end{aligned}$$

This is equivalent to showing that for $B \subset G\gamma_1 \cup G\gamma_2$ the definition of $\mu(B)$ does not depend on the order of covering, i.e.

$$m_{G\gamma_1}(B \cap G\gamma_1) + m_{G\gamma_2}((B \cap G\gamma_2) \backslash G\gamma_1) = m_{G\gamma_2}(B \cap G\gamma_2) + m_{G\gamma_1}((B \cap G\gamma_1) \backslash G\gamma_2).$$

But as

$$m_{G\gamma_1}(B \cap G\gamma_1) = m_{G\gamma_1}(B \cap G\gamma_1 \cap G\gamma_2) + m_{G\gamma_1}((B \cap G\gamma_1) \backslash G\gamma_2)$$

and

$$m_{G\gamma_2}(B \cap G\gamma_2) = m_{G\gamma_2}(B \cap G\gamma_2 \cap G\gamma_1) + m_{G\gamma_2}((B \cap G\gamma_2) \backslash G\gamma_1)$$

we only need to prove that

$$m_{G\gamma_1}(B \cap G\gamma_1 \cap G\gamma_2) = m_{G\gamma_2}(B \cap G\gamma_1 \cap G\gamma_2).$$

Thus we can suppose that $B \subset G\gamma_1 \cap G\gamma_2$. Then we have to prove

Lemma 2. *For $B \subset G\gamma_1 \cap G\gamma_2$ one has*

$$m_{G\gamma_1}(B) = m_{G\gamma_2}(B). \tag{12}$$

Proof. In order to prove this we analyze the bijections Ψ_{γ_1} and Ψ_{γ_2} of B to subsets of $\mathbb{Z}$, for B seen as a subset of $G\gamma_1$ and $G\gamma_2$. We prove that the two subsets of $\mathbb{Z}$ thus obtained are very similar and therefore that their measure m is the same. Let us consider an element of B, i.e. a geodesic $g_0\gamma$. We have then:

$$g_0\gamma = g_0(\gamma\gamma_1^{-1})\gamma_1 = g_0(\gamma\gamma_2^{-1})\gamma_2.$$

Thus,

$$\Psi_{\gamma_1}(g_0\gamma) = \Psi_{\gamma_1}(g_0(\gamma\gamma_1^{-1})\gamma_1) = \psi(\gamma_1\gamma^{-1})$$

and

$$\Psi_{\gamma_2}(g_0\gamma) = \Psi_{\gamma_2}(g_0(\gamma\gamma_2^{-1})\gamma_2) = \psi(\gamma_2\gamma^{-1}).$$

So

$$\begin{aligned}\mathrm{dist}(\Psi_{\gamma_1}(g_0\gamma), \Psi_{\gamma_2}(g_0\gamma)) &= \mathrm{dist}(\psi(\gamma_1\gamma^{-1}), \psi(\gamma_2\gamma^{-1}))\\ &= \mathrm{dist}(\gamma_1\gamma^{-1}, \gamma_2\gamma^{-1})\\ &= \mathrm{dist}(\gamma_1, \gamma_2),\end{aligned}$$

which is independent of γ. This means that the subsets $\{a_i^1\}$, $\{a_i^2\}$ of $\mathbb{Z}$ which correspond to B seen as a subset of $G\gamma_1$ and $G\gamma_2$ satisfy the following:

$$|a_i^1 - a_i^2| = \mathrm{dist}\ (\gamma_1, \gamma_2) = c. \tag{13}$$

We will prove that this suffices to show that

$$m(\{a_i^1\}) = m(\{a_i^2\})$$

as required.

By (13) we have $\{a_i^1\} = D^1 \cup E^1$, $\{a_i^2\} = D^2 \cup E^2$ with $D^1 \cap E^1 = D^2 \cap E^2 = \emptyset$ and

$$D^1 + c = D^2,$$

$$E^1 - c = E^2.$$

Thus by finite additivity of m and its invariance by translations

$$\begin{aligned}m(\{a_i^1\}) &= m(D^1 \cup E^1) = m(D^1) + m(E^1) = m(D^1 + c) + m(E^1 - c)\\ &= m(D^2) + m(E^2) = m(D^2 \cup E^2) = m(\{a_i^2\}).\end{aligned}$$

This ends the proof of Proposition 1. □

Remark. (1) Instead of parameterized geodesics, one could consider unparameterized geodesics. Then it can happen that the set G is finite, i.e.

$$|G| < \infty.$$

But this implies that each $B \in \mathscr{F}$ is finite, i.e.

$$|B| < \infty.$$

Then one can define the measure μ in the following way:

$$\mu(B) = \frac{|B|}{|G|}.$$

This certainly simplifies the proof. So one can ask the following question:

Question 1. *For which groups can we choose a geodesic g_0 in such a way that there are only finitely many different translations of g_0 (different as unparameterized geodesics), containing e?*

(2) In this section we constructed a measure μ on a very special subset $\mathscr{G}$ of all bi-infinite geodesics. It is natural to ask if one can do this for the set of all geodesics, i.e.

Question 2. *Is it possible to construct the measure μ on the set of all bi-infinite geodesics in the group Γ, such that*
(a) μ(geodesics passing through e) $= 1$ *and*
(b) *μ is Γ-invariant?*

(3) In Section 3.2 in order to construct a measure on the subset of geodesics we had to use a translation invariant measure m defined on all subsets of $\mathbb{Z}$. For a subset $\{a_i\}$ of $\mathbb{Z}$, its measure $m(\{a_i\})$ can be defined as

$$m(\{a_i\}) = \lim_{n \to +\infty} \frac{\#\{a_i; a_i \in [-n,n]\}}{2n+1}, \tag{14}$$

where the limit is taken with respect to some ultra-filter ω (the existence of a measure m is equivalent to the existence of a non-trivial ultra-filter). However it is possible to prove the isoperimetric inequality without the use of an ultra-filter. This can be done in the following way. Without going with n to infinity in (14), but for n sufficiently large, we can define a measure on the set of geodesics. This measure is not Γ-invariant. But for a given $A \subset \Gamma$ and for any $\varepsilon > 0$, we can find n large enough so that all equalities in the proof of the isoperimetric inequality hold, up to ε. As ε is arbitrarily small, this proves the desired fact for the subset A, and thus for all subsets of Γ.

(4) Instead of inequality (1) we could consider the following inequality:

$$|A| \leqslant c \sum_{\gamma \in \partial A} (dist(e,\gamma))^d, \tag{15}$$

where c is a positive constant and $0 \leqslant d \leqslant 1$. Inequality (1) shows that (15) holds for any group with $c = d = 1$. But for some groups d can be smaller. For instance, for non-amenable groups (15) holds with $d = 0$. On the other hand, for groups with polynomial growth, d has to be equal to 1. To see this, it is enough to consider the sequence of balls. So for the group Γ let d_Γ be the infimum of d for which (15) holds.

Question 3. *Are there finitely generated groups for which d_Γ is strictly between* 0 *and* 1? *What is d_Γ for a group of intermediate growth constructed in* [4]?

4. Some geometric consequences

Now, we present a consequence of inequality (1) which is due to Sikorav (see [6]).

Let M^n be a compact Riemannian manifold of dimension n and let $\tilde{M}^n$ be its universal cover with the induced Riemannian metric. Let vol denote the volume form on $\tilde{M}^n$. Assume that $\tilde{M}^n$ is not compact. Let us consider a primitive α of the form vol, i.e. $d\alpha =$ vol. In [7] Sullivan defines the notion of a growth of the differential form (see also [5]). As was proved in [6], Theorem 1 has the following consequence:

Theorem 2. *Let M^n be a compact Riemannian manifold whose fundamental group $\pi_1(M^n)$ is infinite. Then there exists a primitive of the volume form on the universal cover $\tilde{M}^n$, which has at most linear growth.*

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